JOURNAL OF MATERIALS SCIENCE 28 (1993) 931–941

Combined weakest link and random defect model for describing strength variability in fibres

W. F. KNOFF
Fibers Research Division, E. I. DuPont De Nemours & Co. Inc., Richmond, Virginia 23261, USA

A mathematical model which describes the strength variability along the length of a fibre was developed. The model is a combination of the modified weakest link and random defect models. This combined model describes very well the strength variability data of aramid fibres.

Nomenclature

L	Specimen length
$F(s)$	Cumulative frequency distribution of link strengths
$1 - F(s)$	Survival function of a link
$F_L(s)$	Cumulative frequency distribution of strengths of a specimen of length L
$1 - F_L(s)$	Survival function of a specimen of length L
s	Strength variable
s_0	Fibre defect-free strength for a random defect or combined model
$s_1, s_2 \ldots$	Fibre strength at the point of a defect
$s'_1, s'_2 \ldots$	Strength a fibre must have at the location of the defect to have a strength of s at the location of the defect
λ	Length of a hypothetical link in a weakest link model
$\rho_1, \rho_2 \ldots$	Defect frequencies (mean number per unit length)
$\nu_1, \nu_2 \ldots$	Defect severities, $0 \leqslant \nu \leqslant 1$
$\rho(s)$	Defect frequency distribution function defined in terms of the strength at the defect
$\xi(\nu)$	Defect frequency distribution function defined in terms of the defect severity
α, β	Defect frequency distribution parameters (Equation 14)
a, b	Weibull distribution parameters (Equation 4)
$P(m)$	Probability that m defects will occur in a given specimen length
m	Number of defects occurring
$\bar{s}$	Mean strength
CV	Coefficient of variation of strength

1. Introduction

Understanding the nature of the variability in tensile strength of fibres and the ability to quantitatively express it in mathematical terms is a subject which has received considerable attention. This effort has been, to a great extent, driven by a need to develop a better understanding of the strength characteristics of structures which are composed of a large number of fibres, such as yarns, cords, fabrics, cables and composite parts. The variability in tensile strength of fibres used in such arrays is an important factor in determining the tensile strength of the final structure relative to the actual average strength of the individual filaments, and thus becomes a key aspect of the design and specification process.

Much theoretical analysis has been carried out in an effort to understand the tensile failure processes of fibre arrays and predict the strength of the array based on the filament properties. Early efforts to address this problem by Daniels [1] and Coleman [2] considered the strength of bundles composed of non-interacting uniform fibres of different strengths. Because of the non-interaction assumption, there was no need to include variability in strength along the length of the fibres. The later inclusion of fibre-to-fibre interaction by which fibres transfer tensile loads among themselves [3, 4] required the consideration of load transfer lengths within the array and therefore some estimation of the magnitude and nature of strength variability along the fibres.

The variability of strength along filaments is also important in other areas of fibre science and technology. Such variability is likely to be a cause of poorer yarn quality and processability in the form of more broken filaments resulting from contact of the filaments with guides and other surfaces. Consideration of strength variability is also important in single-fibre adhesion determination techniques in which the fibre pull-out or segment length is analysed.

There are a number of models available to describe the strength variability on fibres. The choice of the model to be used should primarily depend upon the proposed hypothesis regarding the nature of the strength variability of the particular fibres of interest. The suitability of the proposed model to represent the fibre variability can then be evaluated by comparing predictions of the model with actual data.

2. Earlier models

2.1. Weakest link model (classical and modified)

One mathematical fibre strength model commonly employed to describe fibre strength variability is the classical weakest-link model which is commonly recognized to have been first proposed by Pierce [5]. This model assumes that the length of fibre tested can, with respect to strength, be described as a series of a large number of randomly assembled uniform strength links of which the strengths are independent identically distributed random variables with a common cumulative distribution function. Fig. 1 is a schematic diagram of how the weakest link theory represents fibre strength. The cumulative distribution of the fibre strengths is then given by

$$F_n(s) = 1 - [1 - F(s)]^n \qquad n \gg 1 \tag{1}$$

where $F(s)$ is the common cumulative probability distribution function of the link strengths, s is the strength and n is the number of links needed to describe the fibre. In somewhat different terms, $1 - F_n(s)$ is the probability that a given fibre sample containing n links will be unbroken at an applied force of s. This quantity is more appropriate when discussing a failure and is frequently designated the "survival function".

Further, if λ is the length of the links necessary to describe the fibre and L is the actual length of the specimen, then, if n is very large, $L \gg \lambda$ and L/λ very closely approximates n. Therefore

$$1 - F_L(s) = [1 - F(s)]^{L/\lambda} \qquad L \gg \lambda \tag{2}$$

where $1 - F_L(s)$ is the survival function for a specimen of length L.

It has been shown in a previous paper [6] that the classical weakest–link model has a serious deficiency with respect to describing actual data because it predicts that the measured mean strength of a fibre continues to increase at smaller test lengths, and that this is frequently not the case. In order to provide a better model, a modified weakest–link model has been derived in which there is no restriction regarding the specimen length L with respect to the link length λ. The form of the survival function of fibre strengths for this modified less restrictive form is

$$1 - F_L(s) = [1 - F(s)]^{(L/\lambda)+1} \tag{3}$$

In order for the relationships derived above to be useful, the form of the distribution of link strengths must be available. It is possible to compute the distribution numerically from data. However, this does not reduce the shortcoming of these models and substantially increases the complexity of the necessary computations. The work of Weibull [7] provided a very successful and convenient function form for the distribution of link strengths, which is known as the Weibull distribution:

$$F(s) = 1 - \exp[-(s/a)^b] \quad s \geq 0 \quad a, b > 0 \tag{4a}$$

$$\bar{s} = a\Gamma[1 + (1/b)] \tag{4b}$$

$$CV = \frac{\{\Gamma(1 + 2/b)] - \Gamma^2[1 + (1 + b)]\}^{1/2}}{\Gamma[1 + (1/b)]} \tag{4c}$$

where $\bar{s}$ is the mean, CV is the coefficient of variation and Γ is the gamma function.

Combination of the Weibull distribution with the general classical weakest-link relationship (Equation 2) results in the following:

$$\bar{s} = \lambda^{1/b} a \frac{\Gamma(1 + (1/b)]}{L^{1/b}} \tag{5}$$

Combination with the modified weakest-link relationship (Equation 3) results in the following:

$$\bar{s} = a\frac{\Gamma[1 + (1/b)]}{[(L/\lambda) + 1]^{1/b}} \tag{6}$$

The coefficient of variation of the strength is the same for both the classical and modified weakest-link models. It is identical to Equation 4c and which is only a function of the Weibull parameter, b.

2.2. Random defect model

Another frequently used model is one which is simply described as a random defect model. With respect to the fibre strength, this model considers the specimen to have some constant high level of strength upon which dimensionless non-interacting strength-reducing defects randomly occur. Fig. 2 is a schematic representation of this model. If a single type (severity) of defect exists, then the probability that m defects will occur in a specimen of length L is provided by the Poisson discrete probability distribution function.

$$P_L[m] = \frac{L^m \rho^m \exp(-L\rho)}{m!} \tag{7}$$

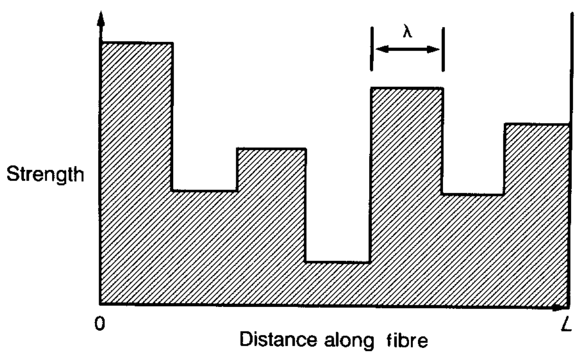

Figure 1 Classical weakest-link model: schematic representation of fibre strength along its length.

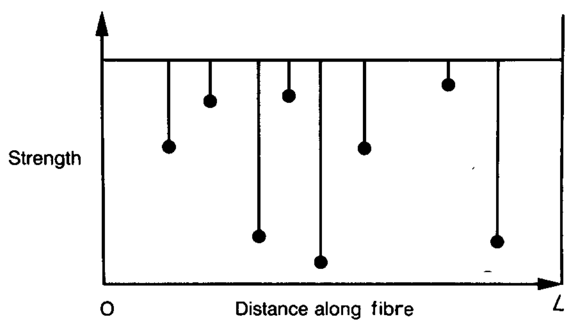

Figure 2 Random defect model: schematic representation of fibre strength along its length.

where $P_L[m]$ is the probability that m defects will occur, ρ is the mean number of defects per unit length, L is the specimen length and m is the number of defects.

For the purpose of defining the strength of this fibre, we are only interested in the case when $m = 0$, i.e. the probability that no defect occurs in a specimen of length L. It is

$$P_L[m = 0] = \exp(-L\rho) \quad (8)$$

If the model fibre has a defect-free strength of s_0 and a strength of s_1 when one or more of the defects of a single type are present, the survival function, $1 - F_L(s)$ is

$$1 - F_L(s) = 1 \qquad s \leqslant s_1 \quad (9a)$$

when the fibre will survive regardless of the presence of a defect,

$$1 - F_L(s) = \exp(-L\rho) \qquad s_1 < s \leqslant s_0 \quad (9b)$$

when the fibre will survive only if no defect is present, and

$$1 - F_L(s) = 0 \qquad s_0 < s \quad (9c)$$

when the fibre will not survive regardless of the absence or presence of a defect. This is shown in Fig. 3.

For n types of defect, designated $1, 2, 3 \ldots i, \ldots n$, which result in fibre strengths $s_n < s_{n-1} < \ldots s_i \ldots < s_2 < s_1 < s_0$, and with mean frequencies $\rho_n, \rho_{n-1} \ldots \rho_i, \ldots \rho_2, \rho_1$, the survival function becomes

$$1 - F_L(s) = 1 \qquad s \leqslant s_n \quad (10a)$$

$$1 - F_L(s) = \exp\left[-L \sum_{j=i+1}^{n} \rho_j\right] \qquad s_{i+1} < s \leqslant s_i$$

$$i = 0, 1, 2 \ldots n-1 \quad (10b)$$

$$1 - F_L(s) = 0 \qquad s_0 < s \quad (10c)$$

The survival function can also be expressed in terms of a continuous frequency distribution of defects;

$$1 - F_L(s) = 1 \qquad s = 0 \quad (11a)$$

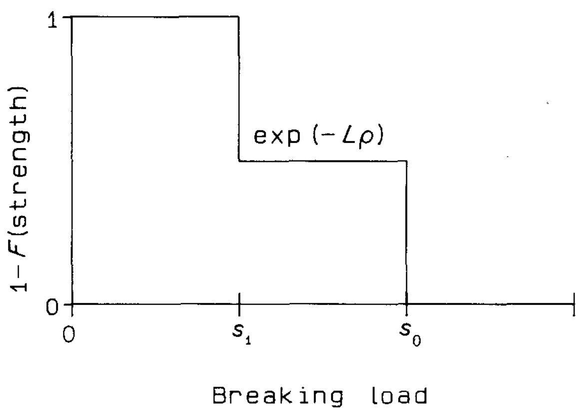

Figure 3 Hypothetical survival function for random defect model with a single type of defect (Equation 9) showing breaking load of fibre if no defect occurs (s_0), and if one or more defects occur (s_1); L = length of specimen, ρ = mean frequency of defects per unit length.

$$1 - F_L(s) = \exp\left[-L \int_0^s \rho(x)\,dx\right] \qquad 0 < s \leqslant s_0 \quad (11b)$$

$$1 - F_L(s) = 0 \qquad s_0 < s \quad (11c)$$

where $\rho(x)$ is the frequency distribution of the defects defined in terms of the fibre strength at the location of the defect. In order to avoid discontinuities in the survival function at $s = s_0$ and $s = 0$, the defect frequency distribution must become unbounded as s approaches s_0 and unity at $s = 0$. The requirement that the defect frequency distribution become unbounded as s approaches s_0 is a result of the fact that a defect ceases to be a defect when $s = s_0$ and therefore it becomes undefined. In many random defect models, the value of s_0 is tacitly assumed to be infinitely large and an appropriate frequency distribution is chosen. A notable example of such a defect frequency distribution is

$$\rho(x) = \frac{bx^{b-1}}{a^b\lambda} \quad (12)$$

which upon substitution into Equation 11b gives

$$1 - F_L(s) = \exp\left[-\frac{L}{\lambda}\left(\frac{s}{a}\right)^b\right] \quad (13)$$

This is the identical expression obtained when the Weibull cumulative distribution function (Equation 4a) is used in the classical weakest-link model expression (Equation 2). This particular example, which was the one chosen by Weibull in his original work, illustrates the fact that the random defect and classical weakest-link models are actually different representations of the same model, a fact which has been noted previously [8]. This is a direct result of the classical weakest-link model requirement that $L \gg \lambda$. The modified weakest-link model, in which there is no restriction as to specimen size L with respect to the link size λ, is a distinct and different model from the classical weakest-link and random defect models.

The form of the defect frequency distribution is of critical importance in defining the applicability and usefulness of this model to real data. A particularly effective defect frequency distribution proposed by Phani [9] is derived from the beta distribution. In its simplest form this is a four-parameter function. However, in the context of the above discussion, two of the parameters are used to define the defect-free strength s_0 and the total number of defects present. The two remaining parameters α and β define the shape of the frequency distribution by the function

$$\int_0^s \rho(x)\,dx = N\frac{(s/s_0)^\alpha}{[1-(s/s_0)]^\beta} \qquad 0 \leqslant s \leqslant s_0 \quad (14)$$

where N is a constant which is related to the number of defects present. This distribution is quite flexible and has been successfully applied to carbon and glass fibre data. However, it is still limited by the fundamental nature of the random defect model. That is, as the mean strength becomes less sensitive to the test length, the coefficient of variation approaches zero.

This property seriously limits the wide application of this model.

3. Combined weakest link and random defect model

The purpose of this paper is to present a model for fibre strength variability which is a combination of both the weakest link and random defect models. With this model the fibre strength is represented as a chain of randomly assembled links of length λ, as in the modified weakest-link model. It is further assumed that these links also contain randomly occurring, independent, non-interacting dimensionless defects which obey the probablistic relationship presented in section 2.2. A schematic representation of this model is shown in Fig. 4.

In order to develop the mathematics of this model, it is best to start with a simple case and build to a generalized expression. Consider a specimen composed of a single link of length λ with a cumulative distribution function of strengths $F(s)$, and that along this specimen two types of non-interacting defect randomly occur. These two types of defects have severities of ν_1 and ν_2 where the severity is defined as the fraction of strength remaining at the location of the defect. Thus, a severity of 0 indicates that no strength remains whereas a severity of 1 indicates that the defect causes no reduction in strength. These defects occur at mean frequencies of ρ_1 and ρ_2. We further define strength s'_1 and s'_2 to be the strengths which a given defect-free link must have such that, if it did contain a defect of severity ν_1 or ν_2, it would have a breaking strength of s. This is clarified mathematically as

$$s'_1 = s/\nu_1 \quad (15a)$$

$$s'_2 = s/\nu_2 \quad (15b)$$

where $s \leqslant s'_1 < s'_2$ and $\nu_2 < \nu_1 \leqslant 1$; Fig. 5 shows the relationships of these strength values with respect to a hypothetical survival function. In the defect frequency distribution above (Equation 14) the ratio s/s_0 is the severity ν as defined in this section.

For this situation, the probability that this specimen is unbroken at an applied force of s is a combination of probabilities regarding the absence of the defects and the defect-free strength of the link. This is the first step in the merging of the mathematics of the random defect and the modified weakest-link models. Consider the conditions necessary in the simple single-link two-defect model described above for the specimen to be unbroken at an applied force s. If s_0 is the defect-free strength of the link of interest and m_1 and m_2 are the numbers of defects of types 1 and 2 respectively occurring in that link, then the total probability P that the link is unbroken at an applied force of s has three terms as shown below:

$$P[\text{specimen unbroken at force } s] = P[m_1 = 0, m_2 = 0]\,P[s \leqslant s_0 < s'_1] + P[m_2 = 0]\,P[s'_1 \leqslant s_0 < s'_2] + P[s'_2 \leqslant s_0] \quad (16)$$

The first term in the above expression addresses the case in which s_0 is greater than the applied force s but less than s'_1. In order for the link to survive a force s, there cannot be a defect of severity either 1 or 2. That is, both m_1 and m_2 must be zero. In the second term, s_0 is greater than s'_1 but less than s'_2. In order for the link to survive an applied force s there cannot be a defect of type 2. That is, m_2 must be zero. The occurrence of a defect of type 1 is of no consequence for this case. The third and last term represents the case in which s_0 is greater than s'_2. If this occurs, the defects are of no consequence. The link will survive regardless of their presence or absence. The probability detailed above is the survival function, $1 - F(s)$, of the specimen and can be transformed using the following relationships. From the random defect model (section 2.2), we obtain

$$P[m_2 = 0] = \exp[-\rho_2\lambda] \quad (17a)$$

$$P[m_1 = 0, m_2 = 0] = \exp[-(\rho_1 + \rho_1)\lambda] \quad (17b)$$

From the weakest link model (section 2.1), we obtain

$$P[s \leqslant s_0 < s'_1] = F(s'_1) - F(s) \quad (18a)$$

$$P[s'_1 \leqslant s < s'_2] = F(s'_1) - F(s'_2) \quad (18b)$$

$$P[s'_2 \leqslant s] = 1 - F(s'_2) \quad (18c)$$

Substituting Equations 17 and 18 into Equation 16,

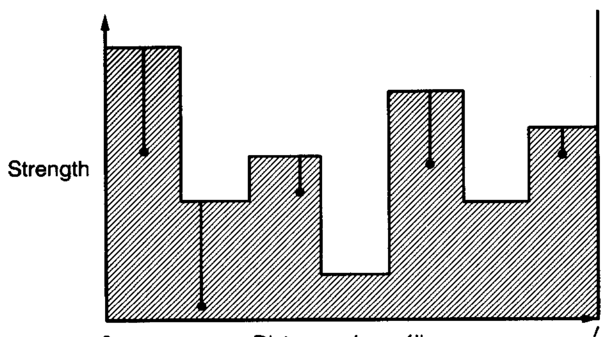

Figure 4 Combined modified weakest link–random defect model: schematic representation of fibre strength along its length.

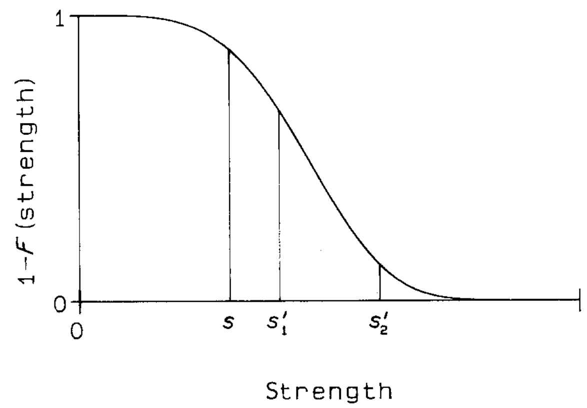

Figure 5 Hypothetical survival function for a single link with two defect severities (Equation 15) showing defect-free strength necessary for link to have strength of s if defect of type 1 is present (s'_1) and if defect of type 2 is present (s'_2).

the following result is obtained:

$$1 - F(s) = \exp[-(\rho_1 + \rho_2)\lambda][F(s'_1) - F(s)] + \exp[-\rho_2\lambda][F(s'_2) - F(s'_1)] + [1 - F(s'_2)] \quad (19)$$

where, again, $F(s)$ is the common cumulative probability distribution function of the link strengths. Generalizing this expression to n types of defect occurring on single link which has been randomly chosen from a population of links with a common cumulative probability distribution $F(s)$, we have

$$\begin{aligned} 1 - F(s) = {} & \exp\left[-\lambda \sum_{i=1}^{n} \rho_i\right][F(s'_1) - F(s)] \\ & + \exp\left[-\lambda \sum_{i=2}^{n} \rho_i\right][F(s'_2) - F(s'_1)] \ldots \\ & + \exp\left[-\lambda \sum_{i=k}^{n} \rho_i\right][F(s'_k) - F(s'_{k-1})] + \ldots \\ & + \exp\left[-\lambda \sum_{i=n}^{n} \rho_i\right][F(s'_n) - F(s'_{n-1})] \\ & + 1 - F(s'_n) \end{aligned} \quad (20)$$

Before this can be expanded to describe a chain of such links by substitution into Equation 3, an important point must be recognized and included in the model. The modified weakest-link model (Equation 3) has no restrictions as to the size of L with respect to λ. It accommodates fractions of links included in the specimen through the assumption that the strength within a link is constant and therefore, the strength introduced by a link is independent of how much of the link actually occurs within the bounds of the specimen. Above, a length dependence has been necessarily introduced through the exponential terms which represent the probability of the absence of the random defects. If Equation 19 were directly substituted into Equation 3, then in the case when $L \ll \lambda$ the computed survival function would actually be that for $L = \lambda$ again because of the exponential terms. This difficulty is solved by substituting the average link length in the specimen, designated Λ, into the exponential terms of Equation 18 and then substituting this into Equation 3. The result is

$$\begin{aligned} 1 - F_L(s) = {} & \left\{\exp\left(-\Lambda \sum_{i=1}^{n} \rho_i\right)[F(s'_1) - F(s)] \right. \\ & + \exp\left(-\Lambda \sum_{i=2}^{n} \rho_i\right)[F(s'_2) - F(s'_1)] \ldots \\ & + \exp\left(-\Lambda \sum_{i=k}^{n} \rho_i\right)[F(s'_k) - F(s'_{k-1})] + \ldots \\ & + \exp\left(-\Lambda \sum_{i=n}^{n} \rho_i\right)[F(s'_n) - F(s'_{n-1})] \\ & \left. + 1 - F(s'_n)\right\}^{(L/\lambda)+1} \end{aligned} \quad (21)$$

In its integral form, this expression is

$$1 - F_L(s) = \left\{\int_s^{\infty} \exp\left[-\Lambda \int_0^{\nu} \xi(x)\,\mathrm{d}x\right] \times \mathrm{d}F\left(\frac{s}{\nu}\right)\mathrm{d}\left(\frac{s}{\nu}\right)\right\}^{(L/\lambda)+1} \quad (22)$$

where ξ is the defect frequency distribution function defined in terms of the defect severity ν, and $\mathrm{d}F$ is the frequency distribution of the link strengths.

For a specimen of length L, which contains $(L/\lambda) + 1$ links represented, the average length of a link segment is

$$\Lambda = \frac{L}{(L/\lambda) + 1} = \frac{\lambda L}{L + \lambda} \quad (23)$$

with the limits

$$L \gg \lambda \qquad \Lambda \simeq \lambda \quad (24a)$$

and

$$L \ll \lambda \qquad \Lambda \simeq L \quad (24b)$$

which are the correct results.

The above relationship (Equation 22) is a generalized expression describing the strength variability along a fibre which includes the effects of both variations in the strength which have finite length, the weakest links, and randomly occurring defects. It collapses to those models at the proper limits. When $\xi(x) = 0$ for all x, that is when there are no random defects present, it becomes

$$\begin{aligned} 1 - F_L(s) & = \left(\int_s^{\infty} \mathrm{d}F\left(\frac{s}{\nu}\right)\mathrm{d}\left(\frac{s}{\nu}\right)\right)^{(L/\lambda)+1} \\ & = [1 - F(s)]^{(L/\lambda)+1} \end{aligned} \quad (25)$$

When the link strength variability is zero and therefore there is no weakest-link character to the model, $\mathrm{d}F(s/\nu)$ becomes a Heaviside unit function at the strength s_0, and the general relationship becomes

$$1 - F_L(s) = \exp\left[-\Lambda \int_0^{s/s_0} \xi(x)\,\mathrm{d}x\right] \quad (26)$$

which in the limit $L \ll \lambda$ when $\Lambda \simeq L$ becomes identical to the survival function for the random defect model (Equation 11).

4. Experimental procedure

The strength (breaking load) of single filaments is determined by a technique which has been developed in our laboratory. All filaments are mounted on paper tabs with an amine catalysed cyanoacrylate adhesive. We have found this mounting technique to be quite satisfactory in terms of reproducibility, convenience and mounting speed. Tensile testing is carried out on an Instron Model 1122 tester equipped with 500 g capacity pneumatic grips. The elongation rate was approximately 20% $\min^{-1}$.

In order for the strength data to be suitable for analysis and useful for testing strength-variability models, the sampling and testing must be done in a specific manner. Of course, the analysis must be done at various test lengths. For each test length chosen several filaments were randomly sampled from the yarn bundle and multiple strength determinations conducted along each filament. Data collected in this way can then be analysed to determine both the interfilament and intrafilament components of strength variability. The importance of this is discussed below.

On all figures, the error bars designate the 95% confidence limits of that value.

Five Kevlar® aramid fibre samples are included in this study. Three samples (designated Item B, Item C and Item D) are commercial-quality fibres. The data from Items B and C have been reported and discussed previously [6]. Samples E1 and E2 are a pair of experimental samples which are of particular interest for modelling. Both were manufactured at the same time. Item E2, however, was made by a process specifically designed to eliminate strength-reducing defects. The data summary tables (Tables I to V) contain the pertinent information regarding each of these samples. The data from Items E1 and E2 were provided by Dr T. S. Chern.

TABLE I Filament breaking load data summary, Item B

Parameter	Test length (cm)				
	0.18	1.0	2.5	5.1	25.4
Strength					
Mean (dN)	4.75	4.68	4.39	4.15	3.48
Standard error (dN)	0.15	0.08	0.11	0.09	0.12
Determinations, total	80	80	56	56	56
Filaments tested	8	8	8	8	8
CV (%)					
Total	12.4	16.1	17	18.8	26.5
Within filament[a]	9.8	9.3	11.9	11.9	25.2

[a] From joint estimate [10].

TABLE II Filament breaking load data summary, Item C

Parameter	Test length (cm)				
	0.18	1.0	2.5	5.1	25.4
Strength					
Mean (dN)	4.84	4.8	4.71	4.39	3.81
Standard error (dN)	0.05	0.06	0.05	0.09	0.1
Determinations, total	80	80	56	56	56
Filaments tested	8	8	8	8	8
CV (%)					
Total	8.5	6.7	8.1	10.7	16
Within filament[a]	8.4	6.1	7.9	9.9	15.3

[a] From joint estimate [10].

TABLE III Filament breaking load data summary, Item D

Parameter	Test length (cm)				
	0.2	1.0	2.5	12.7	25.4
Strength					
Mean (dN)	4.41	4.44	4.2	3.18	2.87
Standard error (dN)	0.06	0.1	0.11	0.12	0.12
Determinations, total	312	100	100	100	100
Filaments tested	36	10	10	10	10
CV (%)					
Total	12.2	12.4	17.7	22.5	26.6
Within filament[a]	9.7	10.9	16.8	20.7	25.2

[a] From joint estimate [10].

TABLE IV Filament breaking load data summary, Item E1

Parameter	Test length (cm)		
	0.18	2.5	25.4
Strength			
Mean (dN)	4.88	4.47	3.66
Standard error (dN)	0.1	0.14	0.26
Determinations, total	24	24	24
Filaments tested	4	4	4
CV (%)			
Total	9.4	15	28.1
Within filament[a]	9.4	15	28.1

[a] From joint estimate [10].

TABLE V Filament breaking load data summary, Item E2

Parameter	Test length (cm)		
	0.18	2.5	25.4
Strength			
Mean (dN)	5.19	5.01	4.51
Standard error (dN)	0.14	0.14	0.12
Determinations, total	24	24	24
Filaments tested	4	4	4
CV (%)			
Total	8.4	10.3	9.3
Within filament[a]	7.3	9.7	8.6

[a] From joint estimate [10].

5. Discussion

5.1. Data reduction and analysis

The model deals only with a distribution of link strengths along a single hypothetical filament. It ignores any differences in the distribution between filaments and, therefore, is strictly applicable to data collected from either a sample in which all filaments have the same distributions of strengths along their length or a sample of a single filament which has a uniform distribution of strengths along its length. In practice, neither of these situations is likely or very practical and, therefore, sources of overall variability within a sample must be recognized and only those components pertinent to the model considered.

Only the within-filament component of strength variability should be used when applying the filament strength versus test length data presented here to the model. A distribution in mean strength between filaments (the between-filament component of strength variability) will contribute to the overall strength coefficient of variation in the yarn sample, but will not impact the response of average strength of coefficient of variation to test length. This fact is clear if one considers a yarn sample composed of perfectly uniform filaments of different strengths. For this hypothetical sample, the measured mean and coefficient of variation would be independent of test length.

A joint estimate of the within-filament strength coefficient of variation at each test length was made [10].

If the within-filament variances constitute a homogeneous population, then this value is an estimate of the common within-filament coefficient of variation and the conditions of the model are well satisfied. If the variances are not homogeneous, the value is an estimate of the average within-filament strength coefficient of variation. This is a departure from the assumptions of the model and, depending on the degree of non-homogeneity, could cause significant differences between the predicted and observed response of strength to test length. Bartlett's chi-square statistic indicates that, in most cases, the null hypothesis of variance equivalency can be rejected with a reasonable degree of confidence. The confidence of this rejection, however, may be erroneously high because of the assumption of normality made in the chi-square statistic test. Testing of the variance homogeneity using the gamma plotting technique [11], which is less sensitive to deviations from normality than is the chi-square statistic, does not support the rejection of the null hypothesis of variance equivalency. Therefore, it is valid to assume that the jointly estimated coefficient of variation is an estimate of the common within-filament coefficient of variation.

5.2. Application of models

In general, the weakest link models, both classical and modified, and the random defect model are unable to describe well the nature of fibre strength variability with respect to the first and second moment of the strength distribution. As with any modelling effort, if the model fails to describe the observations, then one must conclude that the perception of reality which the model attempts to capture is not sufficiently correct.

The strengths and deficiencies of the weakest link models have been mentioned above and discussed in a previous publication [6]. They are reviewed here using the Item B data (Table I) as an example. The important improvement which the modified model provides over the classical model is that the modified model can, if necessary, represent in a physically meaningful manner a plateau or levelling of the mean strength at lower test lengths. The classical model cannot (Fig. 6), a deficiency which has been noted by others [12]. Both weakest link models, when formulated with a Weibull distribution of link strengths, result in no change in the coefficient of variation with the test length. This is generally not true and is specifically not true for Item B (Fig. 7). The significant increase in the coefficient of variation at the higher test length indicates a breakdown of one or more of the assumptions made in the application of these models. Three possible causes for the observed increase in coefficients of variation are non-homogeneity of the within-filament link strength distributions from filament to filament, a deviation of the link strength distribution from a Weibull in the low-strength tail region, and the influence of random dimensionless defects which are not represented by these models. The second reason above has been addressed, and some investigation of the benefit of using a numerically defined link strength distribution has been conducted with little success. This lead to the effort to develop a model which incorporates the effect of both finite-length links and random dimensionless defects.

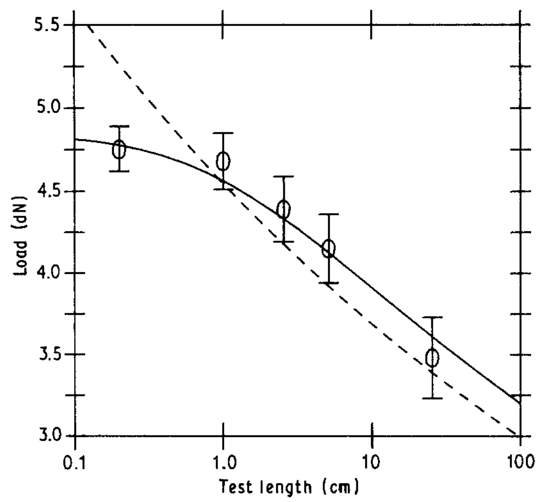

Figure 6 Mean filament breaking load as a function of test length for Item B. (○) Table I; (——) modified weakest link (Equation 6) with $a = 5.1$ dN, link strength $CV = 10.8\%$ ($b = 11$) and $\lambda = 1.0$ cm; (- - -) classical weakest link (Equation 5) with $\lambda^{(1/b)} a\Gamma\,[1 + (1/b)] = 4.55$ dN, link strength $CV = 10.8\%$ ($b = 11$).

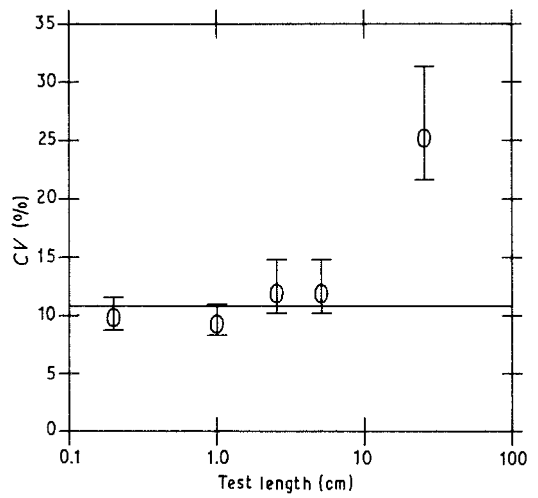

Figure 7 Within-filament CV as a function of test length for Item B. (○) Table I; (——) classical and modified weakest-link with Weibull distribution of link strengths and link strength $CV = 10.8\%$ ($b = 11$).

As has been mentioned, the suitability of the random defect model is critically dependent upon the form of the defect frequency distribution employed, and perhaps the best distribution function yet identified is one derived from the beta distribution. This was described in section 2.2 above (Equation 14). Because of the required analysis of the raw data to remove the component of variability introduced by the between-filament strength variation (section 5.1), a fit of this random defect model cannot be obtained via conventional probability plotting techniques. Fitting was, therefore, carried out by means of a numerical "cut and try" approach. Although this technique admittedly does not produce an absolute best fit, it is close and

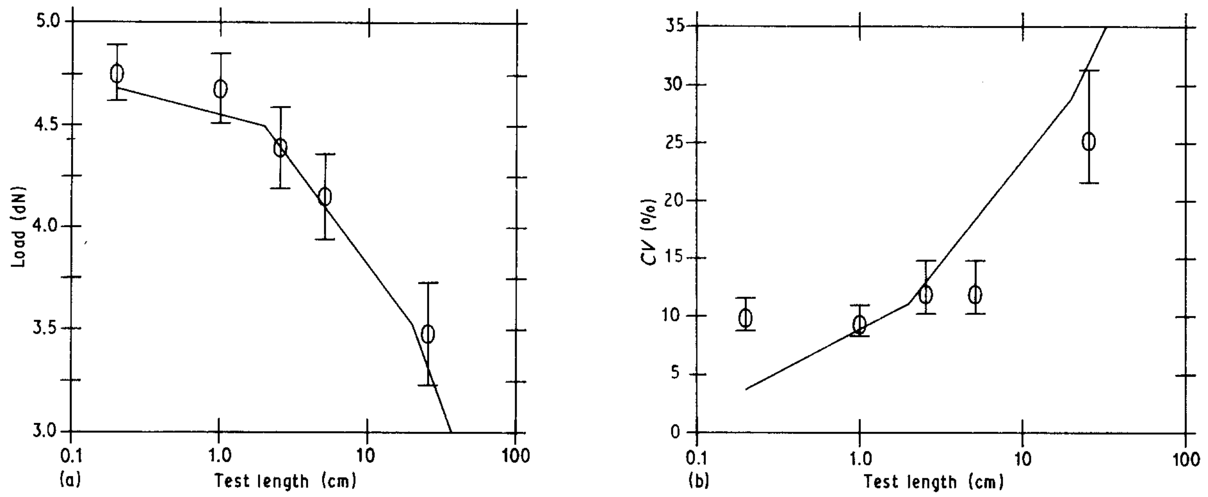

Figure 8 (a) Mean filament breaking load and (b) within-filament CV as functions of test length for Item B. (○) Table I; (——) random defect model (Equation 14 defect frequency distribution) with $\alpha = 1$, $\beta = 1$, $s_0 = 4.7$ dN and 1.25 defects cm^{-1}.

effectively demonstrates the capabilities and limitations of this model. The unbounded nature of Equation 14 as s/s_0 approaches unity results in a computational problem which is avoided by truncating the function at $s/s_0 = 0.99$. That is, for the computational purposes of this paper, only defects which result in a 1% or greater reduction in strength are considered ($0 \leqslant s/s_0 \leqslant 0.99$). The numerous very minor defects which result in less than 1% strength reduction ($0.99 < s/s_0 \leqslant 1$) are neglected. The defect frequency quoted in the following examples is that for defects which result in a strength reduction of 1% or more.

The wider applicability of this distribution (Equation 14) has been further tested by attempting a fit to the data from Item B. A fit was first attempted by focusing on the mean strength data. A good fit was possible with the distribution shape parameters $\alpha = 1$ and $\beta = 1$ (Fig. 8a). However, these values resulted in an unsatisfactory fit for the CV data (Fig. 8b). When the best fit of the CV data is obtained with $\alpha = 0.1$ and $\beta = 10$ (Fig. 9a), the fit of the mean breaking load data is poor (Fig. 9b). These examples demonstrate how the correlation of the change in mean breaking load with test length and CV causes difficulties in applying this model to real data. This correlation is an unavoidable consequence of this model.

The combined random defect–weakest link model described overcomes the limitations of the other models discussed above. The fitting of this model to the data has not yet been adapted to a totally computational approach, and therefore a methodology has been used which is guided by the understanding of how the various model parameters influence the results. Firstly, it should be noted that if the data can be fitted using one of the simpler models described (random defect or modified weakest link) with little to be gained by going to the more complex combined model, then there needs to be some additional knowledge which would justify using the combined model. This could be diameter data which indicate significant variation over fibre lengths comparable to the test length. In the case of Item B, which we have been using as an example, the use of the combined model appears appropriate.

Fitting the combined model to Item B and the subsequent examples was done as follows. For this fitting exercise the defect frequency distribution parameters α and β were both assumed to be equal to

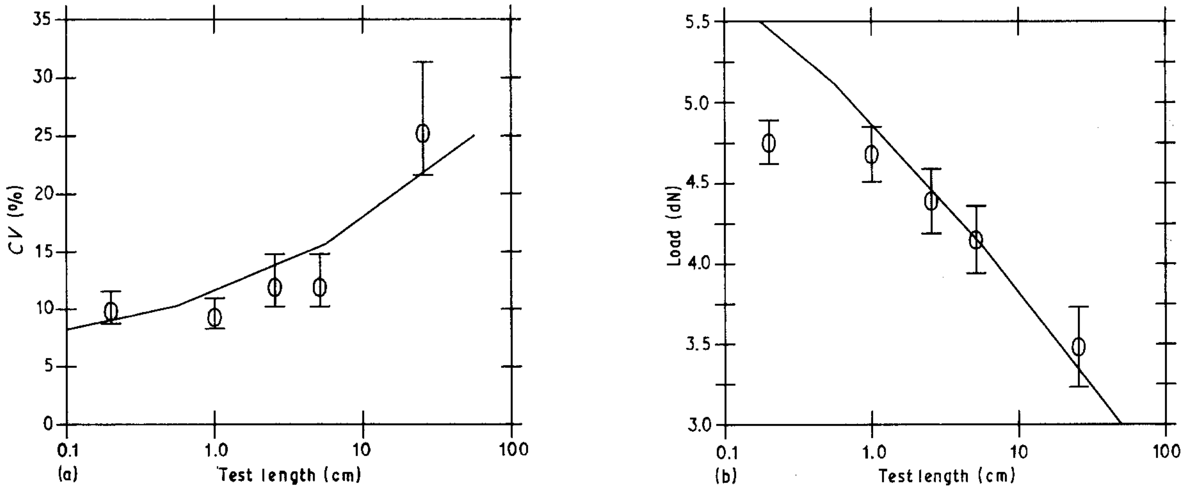

Figure 9 (a) Within-filament CV and (b) mean filament breaking load as functions of test length for Item B. (○) Table I; (——) random defect model (Equation 14 defect frequency distribution) with $\alpha = 0.1$, $\beta = 10$, $s_0 = 8.8$ dN and 1.8×10^{17} defects cm^{-1}.

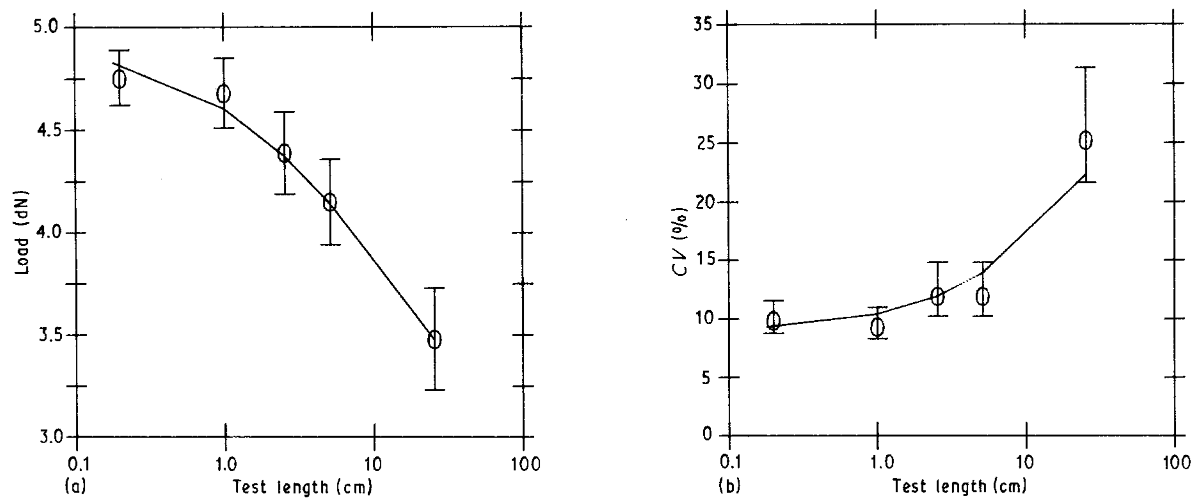

Figure 10 (a) Mean filament breaking load and (b) within-filament CV as functions of test length for Item B. (○) Table I; (——) combined model with $\alpha = 1$, $\beta = 1$, 0.6 defects cm^{-1}, $s_0 = 4.95$ dN, link $CV = 9\%$ and $\lambda = 1$ cm.

unity. Although this reduced the capability of the model, it made the fitting procedure more tractable and also permitted direct comparison of the defect frequency values determined for different samples. If different parameters α and β are used, the cases cannot be compared simply based on the total number of defects present, because the shape of the distribution has changed. The coefficient of variation and mean strength of the Weibull distribution of links were estimated from the asymptotic coefficient of variation and mean maximum strength occurring at the lower test lengths. These were 9% and 4.95 dN, respectively. The link length λ was initially set to some large value, approximately 1000 cm, such that its influence on the fitted function would initially be insignificant, and then the number of defects present (1% or greater reduction of strength) was adjusted to obtain the best possible fit to the coefficient of variation data. Then, the value of λ was adjusted to obtain the best fit of the mean strength versus test length data. This procedure was repeated and the parameters adjusted to get the best visual fit. As can be seen in Fig. 10, the fit obtained for Item B is quite good and clearly superior to the fits attainable with either the weakest link or random defect models. The approximate best fit was achieved for $\lambda = 1$ cm, $s_0 = 4.95$ dN and $CV = 9\%$ with an average of 0.6 defects cm^{-1} present ($\alpha = 1$ and $\beta = 1$).

The above procedure was also applied to the data from Item C (Fig. 11) and Item D (Fig. 12). As can be seen from the fits, the combined model very well describes the data from these samples and demonstrates the general usefulness of the combined model for describing the strength variability of aramid fibres.

Of more specific interest are the data from items E1 and E2. As was described, these items were both manufactured at the same time, but item E2 was made using a process which was specifically designed to minimize strength-reducing defects. Because of this, one might speculate that the weakest-link component of the strength variability for these samples, possibly associated with denier variability, may be similar, but that the random-defect component may be quite different. Examination of the data from both items reveals that the CV of strength of item E2 does not depend on test length. This is a characteristic of the

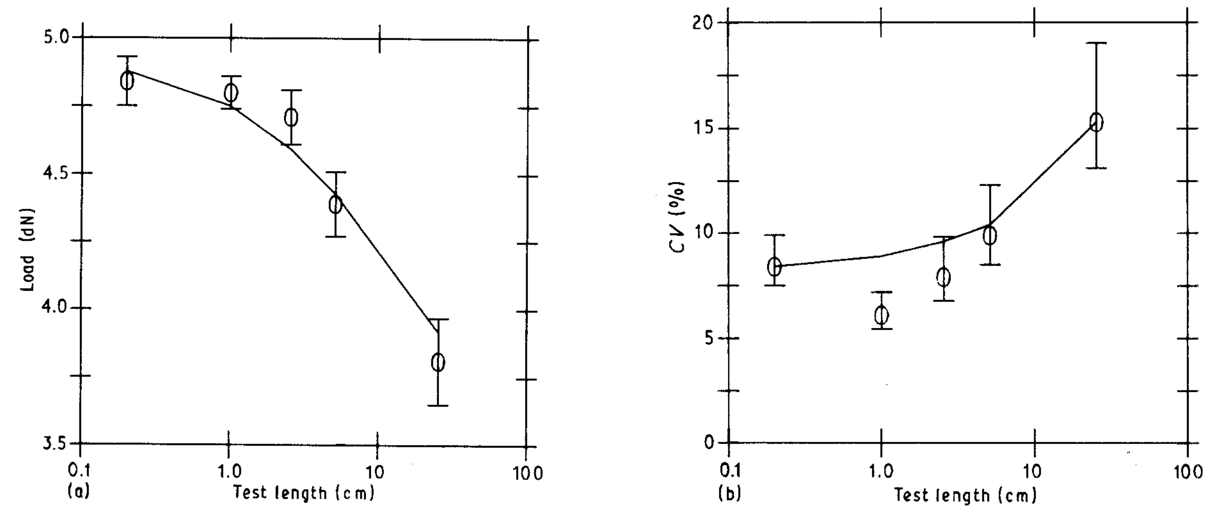

Figure 11 (a) Mean filament breaking load and (b) within-filament CV as functions of test length for Item C. (○) Table II; (——) combined model with $\alpha = 1$, $\beta = 1$, 0.2 defects cm^{-1}, $s_0 = 4.96$ dN, link $CV = 8.2\%$ and $\lambda = 1.75$ cm.

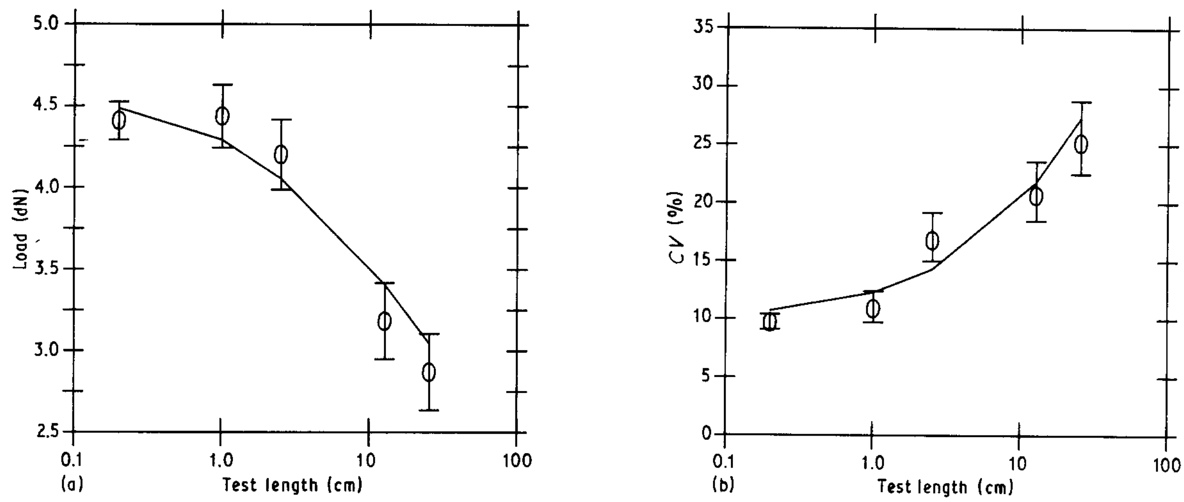

Figure 12 (a) Mean filament breaking load and (b) within-filament CV as functions of test length for Item D. (○) Table II; (——) combined model with $\alpha = 1$, $\beta = 1$, 1.0 defect cm^{-1}, $s_0 = 4.60$ dN, link $CV = 10.0\%$ and $\lambda = 1.5$ cm.

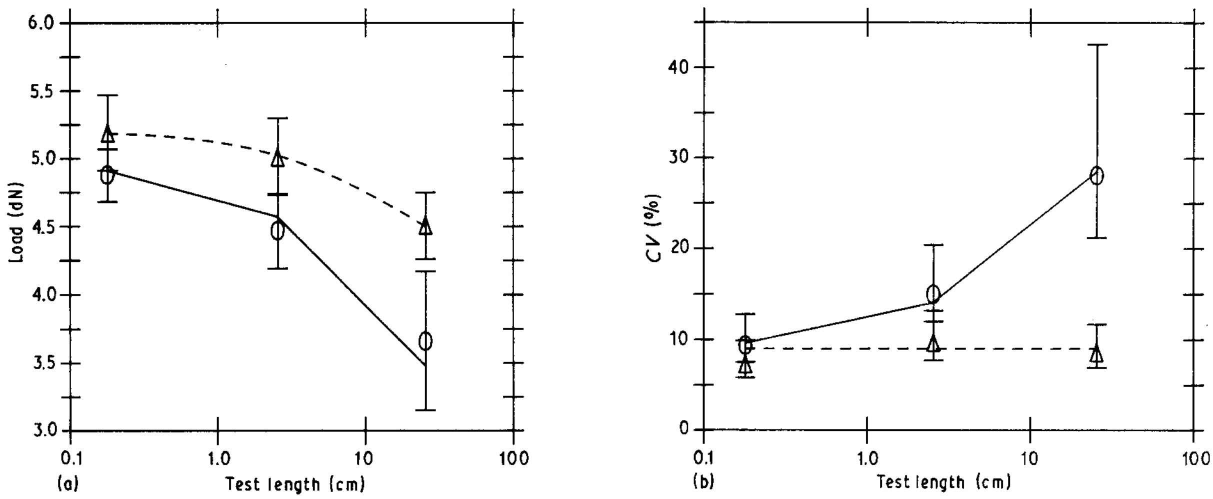

Figure 13 (a) Mean filament breaking load and (b) within-filament CV as functions of test length for Items E1 and E2. (○) Table IV (E1); (——) combined model with $\alpha = 1$, $\beta = 1$, 1.0 defect cm^{-1}, $s_0 = 5.0$ dN, link $CV = 9.0\%$ and $\lambda = 4.2$ cm. (△) Table V (E2); (- - -) modified weakest-link model (no defects) with $s_0 = 5.2$ dN, link $CV = 9.0\%$ and $\lambda = 4.2$ cm.

modified weakest-link model in which no random defects are considered (section 2.1). A regression fit of the Item E2 mean breaking load data to the modified weakest-link relationship (Equation 6) produced a very satisfactory fit with $\lambda = 4.2$ cm, $s_0 = 5.2$ dN and $CV = 9\%$ (Fig. 13a). The CV of strength of Item E1 shows the increase at higher test lengths which is characteristic of random defects and suggests that the combined random defect–modified weakest-link model is appropriate. A very good fit of the Item E1 data was achieved by computationally simply adding an average of 1.0 defects cm^{-1} to the modified weakest-link model fibre used for the Item E2 data (with a small adjustment of s_0 from 5.2 to 5.0 dN). The fits obtained are shown (Fig. 13). The results of fitting these data are quite significant in that they support that validity of this modelling approach and demonstrate its ability to resolve the different nature of strength variability in two aramid fibre samples.

The parameters used for the best fit of the aramid fibre samples described above have been summarized in Table VI.

TABLE VI Aramid fibre breaking-load variability parameters–combined modified weakest link–random defect model

Parameter	Item				
	B	C	D	E1	E2
Link length (cm)	1	1.75	1.5	4.2	4.2
CV of link strength (%)	9	8.2	10	9	9
Mean link strength (dN)[a]	4.76	4.79	4.41	4.81	5
Defects cm^{-1}[b]	0.6	0.2	1	1	0

[a] Equation 4b.
[b] Number of defects resulting in 1% or greater reduction in strength.

6. Conclusions

Available statistical models for describing the strength variability along the length of a fibre existing prior to this work were the classical weakest-link or random defect models and the modified weakest-link model. These models have been observed to be inadequate for describing real data and specifically, in this study, not able to statisfactorily describe strength data for several Kevlar® aramid fibre samples. It is possible to mathematically describe a model in which the fibre, with respect to its strength, is considered to be a chain of randomly assembled finite-length links possessing a common strength distribution function on which strength-reducing dimensionless defects randomly occur. That is, a combination of the modified weakest-link and random defect models.

The combined model is able to describe the aramid fibre data quite well and considerably better than either of the single-mode models. The fitting parameters needed to obtain good fits (Table VI) show that, in general, the strength variability along the length of the Kevlar® aramid fibres examined in this paper is characterized by 1.0 to 4.0 cm long lengths of relatively constant strength (with a CV of 8 to 10% between the strength of these lengths) on which an average of 0 to 1 defects cm^{-1} randomly occur.

The type of information gained from this modelling can be powerful in understanding the sources and impact of fibre strength variability. Although this model has been specifically developed for Kevlar® aramid fibres, it is very likely to be applicable to other fibres because it occupies such a unique position in combining the statistics of modified weakest-link and random defect models.

Acknowledgements
Sample preparation and testing was ably and professionally carried out by Mr Andrew Byrance. Dr T. S. Chern provided the data for Items E1 and E2.

References
1. H. E. DANIELS, *Proc. R. Soc.* **83A** (1945) 405.
2. B. D. COLEMAN, *J. Mech. Phys. Solids* **7** (1958) 60.
3. W. B. ROSEN, *American Institute of Aeronautics and Astronautics Journal* **2** (1964) 1985.
4. C. ZWEBEN, *ibid.* **6** (1968) 2325.
5. F. T. S. PEIRCE, *J. Text. Inst.* **17** (1926) 355.
6. W. F. KNOFF, *J. Mater. Sci.* **22** (1987) 1024.
7. W. WEIBULL, "Handlingar", No. 151 (Royal Swedish Academy of Energy Sciences, 1939).
8. S. L. PHEONIX, in "Composite Materials: Testing and Design (Third Conference)", ASTM STP 546 (American Society for Testing and Materials, Philadelphia, 1974) p. 130.
9. K. K. PHANI, *Compos. Sci. Technol.* **30** (1987) 59.
10. C. A. BENNETT and N. L. FRANKLIN, "Statistical Analysis in Chemistry and Chemical Industry" (Wiley, New York, 1954) pp. 319–321.
11. M. B. WILK and R. GANANDESIKANN, *Biometrica* **55** (1968) 1.
12. A. S. WATSON and R. L. SMITH, *J. Mater. Sci.* **20** (1985) 3260.

Received 26 July 1991
and accepted 4 February 1992